MATHEMATICS
FORMULA BOOKLET - GYAAN SUTRA

INDEX

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STRAIGHT LINE

1. Distance Formula:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

2. Section Formula :

$$x = \frac{mx_2 + nx_1}{m+n} \; ; \; y = \frac{my_2 + ny_1}{m+n}.$$

3. Centroid, Incentre & Excentre:

Centroid $G \equiv \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$,

Incentre $I \equiv \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$

Excentre $I_1 \equiv \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$

4. Area of a Triangle:

$$\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

5. Slope Formula:

Line Joining two points $(x_1\, y_1)$ & $(x_2\, y_2)$, $m = \frac{y_1 - y_2}{x_1 - x_2}$

6. Condition of collinearity of three points:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

7. Angle between two straight lines :

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

8. Two Lines :

$ax + by + c = 0$ and $a'x + b'y + c' = 0$ two lines

1. parallel if $\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$.

2. Distance between two parallel lines = $\left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$.

3 Perpendicular : If $aa' + bb' = 0$.

9. A point and line:

1. Distance between point and line = $\left|\frac{a x_1 + b y_1 + c}{\sqrt{a^2 + b^2}}\right|$.

2. Reflection of a point about a line:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -2\frac{a x_1 + b y_1 + c}{a^2 + b^2}$$

3. Foot of the perpendicular from a point on the line is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\frac{a x_1 + b y_1 + c}{a^2 + b^2}$$

10. Bisectors of the angles between two lines:

$$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = \pm \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$$

11. Condition of Concurrency :

of three straight lines $a_i x + b_i y + c_i = 0$, $i = 1,2,3$ is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

12. A Pair of straight lines through origin:

$ax^2 + 2hxy + by^2 = 0$

If θ is the acute angle between the pair of straight lines, then $\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$.

CIRCLE

1. Intercepts made by Circle $x^2 + y^2 + 2gx + 2fy + c = 0$ **on the Axes:**

(a) $2\sqrt{g^2 - c}$ on x -axis (b) $2\sqrt{f^2 - c}$ on y - aixs

2. Parametric Equations of a Circle:

$x = h + r\cos\theta$; $y = k + r\sin\theta$

3. Tangent :

(a) Slope form : $y = mx \pm a\sqrt{1 + m^2}$

(b) Point form : $xx_1 + yy_1 = a^2$ or T = o

(c) Parametric form : $x\cos\alpha + y\sin\alpha = a$.

4. Pair of Tangents from a Point: $SS_1 = T^2$.

5. Length of a Tangent : Length of tangent is $\sqrt{S_1}$

6. Director Circle: $x^2 + y^2 = 2a^2$ for $x^2 + y^2 = a^2$

7. Chord of Contact: T = 0

1. Length of chord of contact = $\dfrac{2LR}{\sqrt{R^2 + L^2}}$

2. Area of the triangle formed by the pair of the tangents & its chord of contact = $\dfrac{RL^3}{R^2 + L^2}$

3. Tangent of the angle between the pair of tangents from (x_1, y_1)

$$= \left(\frac{2RL}{L^2 - R^2}\right)$$

4. Equation of the circle circumscribing the triangle PT_1T_2 is :

$$(x - x_1)(x + g) + (y - y_1)(y + f) = 0.$$

8. Condition of orthogonality of Two Circles: $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

9. Radical Axis : $S_1 - S_2 = 0$ i.e. $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$.

10. Family of Circles: $S_1 + KS_2 = 0$, $S + KL = 0$.

PARABOLA

1. Equation of standard parabola :

$y^2 = 4ax$, Vertex is (0, 0), focus is (a, 0), Directrix is $x + a = 0$ and Axis is $y = 0$. Length of the latus rectum = 4a, ends of the latus rectum are L(a, 2a) & L' (a, − 2a).

2. Parametric Representation: $x = at^2$ & $y = 2at$

3. Tangents to the Parabola $y^2 = 4ax$:

1. Slope form $y = mx + \frac{a}{m}$ $(m \neq 0)$ 2. Parametric form $ty = x + at^2$
3. Point form T = 0

4. Normals to the parabola $y^2 = 4ax$:

$y - y_1 = -\frac{y_1}{2a}(x - x_1)$ at (x_1, y_1) ; $y = mx - 2am - am^3$ at $(am^2, -2am)$;

$y + tx = 2at + at^3$ at $(at^2, 2at)$.

ELLIPSE

1. Standard Equation : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$ & $b^2 = a^2(1 - e^2)$.

Eccentricity: $e = \sqrt{1 - \frac{b^2}{a^2}}$, $(0 < e < 1)$, **Directrices :** $x = \pm \frac{a}{e}$.

Focii : $S \equiv (\pm ae, 0)$. Length of, major axes = 2a and minor axes = 2b

Vertices : $A' \equiv (-a, 0)$ & $A \equiv (a, 0)$.

Latus Rectum : $= \frac{2b^2}{a} = 2a(1 - e^2)$

2. Auxiliary Circle : $x^2 + y^2 = a^2$

3. Parametric Representation : $x = a \cos\theta$ & $y = b \sin\theta$

4. Position of a Point w.r.t. an Ellipse:

The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as;

$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > < \text{ or } = 0$.

5. **Line and an Ellipse:**

The line y = mx + c meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two points real, coincident or imaginary according as c^2 is < = or > $a^2m^2 + b^2$.

6. **Tangents:**

Slope form: $y = mx \pm \sqrt{a^2m^2 + b^2}$, Point form : $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$,

Parametric form: $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

7. **Normals:**

$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$, $ax.\sec\theta - by.\operatorname{cosec}\theta = (a^2 - b^2)$, $y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$.

8. **Director Circle:** $x^2 + y^2 = a^2 + b^2$

HYPERBOLA

1. **Standard Equation:**

Standard equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2 (e^2 - 1)$.

Focii : $S \equiv (\pm ae, 0)$ **Directrices :** $x = \pm \frac{a}{e}$

Vertices : $A \equiv (\pm a, 0)$

Latus Rectum (ℓ) : $\ell = \frac{2b^2}{a} = 2a (e^2 - 1)$.

2. **Conjugate Hyperbola :**

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are conjugate hyperbolas of each.

3. **Auxiliary Circle :** $x^2 + y^2 = a^2$.

4. **Parametric Representation :** $x = a \sec\theta$ & $y = b \tan\theta$

5. Position of A Point 'P' w.r.t. A Hyperbola :

$S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 >, = \text{ or } < 0$ according as the point (x_1, y_1) lies inside, on or outside the curve.

6. Tangents :

(i) Slope Form : $y = mx \pm \sqrt{a^2m^2 - b^2}$

(ii) Point Form : at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

(iii) Parametric Form : $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

7. Normals :

(a) at the point P (x_1, y_1) is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 = a^2e^2$.

(b) at the point P $(a\sec\theta, b\tan\theta)$ is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2$.

(c) Equation of normals in terms of its slope 'm' are $y = mx \mp \frac{(a^2 + b^2)m}{\sqrt{a^2 - b^2m^2}}$.

8. Asymptotes : $\frac{x}{a} - \frac{y}{b} = 0$ and $\frac{x}{a} + \frac{y}{b} = 0$.

Pair of asymptotes : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

9. Rectangular Or Equilateral Hyperbola : $xy = c^2$, eccentricity is $\sqrt{2}$.

Vertices : $(\pm c, \pm c)$; Foci : $(\pm\sqrt{2}c, \pm\sqrt{2}c)$. Directrices : $x + y = \pm\sqrt{2}c$

Latus Rectum (l) : $l = 2\sqrt{2}\,c$ = T.A. = C.A.

Parametric equation $x = ct$, $y = c/t$, $t \in R - \{0\}$

Equation of the tangent at P (x_1, y_1) is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ & at P(t) is $\frac{x}{t} + ty = 2c$.

Equation of the normal at P (t) is $xt^3 - yt = c(t^4 - 1)$.

Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

LIMIT OF FUNCTION

1. Limit of a function f(x) is said to exist as $x \to a$ when,

$$\underset{h \to 0^+}{\text{Limit}}\, f(a - h) = \underset{h \to 0^+}{\text{Limit}}\, f(a + h) = \text{some finite value M.}$$

(Left hand limit) (Right hand limit)

2. Indeterminant Forms:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, \infty^0, 0^0, \text{ and } 1^\infty.$$

3. Standard Limits:

$$\underset{x \to 0}{\text{Limit}} \frac{\sin x}{x} = \underset{x \to 0}{\text{Limit}} \frac{\tan x}{x} = \underset{x \to 0}{\text{Limit}} \frac{\tan^{-1} x}{x}$$

$$= \underset{x \to 0}{\text{Limit}} \frac{\sin^{-1} x}{x} = \underset{x \to 0}{\text{Limit}} \frac{e^x - 1}{x} = \underset{x \to 0}{\text{Limit}} \frac{\ell n(1 + x)}{x} = 1$$

$$\underset{x \to 0}{\text{Limit}}\, (1 + x)^{1/x} = \underset{x \to \infty}{\text{Limit}} \left(1 + \frac{1}{x}\right)^x = e, \quad \underset{x \to 0}{\text{Limit}} \frac{a^x - 1}{x} = \log_e a, \ a > 0,$$

$$\underset{x \to a}{\text{Limit}} \frac{x^n - a^n}{x - a} = na^{n-1}.$$

4. Limits Using Expansion

(i) $a^x = 1 + \frac{x \ln a}{1!} + \frac{x^2 \ln^2 a}{2!} + \frac{x^3 \ln^3 a}{3!} + \ldots\ldots\ldots a > 0$

(ii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

(iii) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots \text{for } -1 < x \le 1$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots$

(v) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$

(vi) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots$

(vii) for $|x| < 1$, $n \in R$ $(1 + x)^n$

$$= 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots\ldots\ldots\ldots$$

5. Limits of form 1^∞, 0^0, ∞^0

Also for $(1)^\infty$ type of problems we can use following rules.

$\lim_{x\to 0}(1 + x)^{1/x} = e$, $\lim_{x\to a}[f(x)]^{g(x)}$,

where $f(x) \to 1$; $g(x) \to \infty$ as $x \to a$ = $e^{\lim_{x\to a}[f(x)-1]g(x)}$

6. Sandwich Theorem or Squeeze Play Theorem:

If $f(x) \le g(x) \le h(x)$ $\forall$ x & $\underset{x\to a}{\text{Limit}}\, f(x) = \ell = \underset{x\to a}{\text{Limit}}\, h(x)$ then $\underset{x\to a}{\text{Limit}}\, g(x) = \ell$.

METHOD OF DIFFERENTIATION

1. Differentiation of some elementary functions

1. $\frac{d}{dx}(x^n) = nx^{n-1}$

2. $\frac{d}{dx}(a^x) = a^x \ell n\, a$

3. $\frac{d}{dx}(\ell n|x|) = \frac{1}{x}$

4. $\frac{d}{dx}(\log_a x) = \frac{1}{x\ell n a}$

5. $\frac{d}{dx}(\sin x) = \cos x$

6. $\frac{d}{dx}(\cos x) = -\sin x$

7. $\frac{d}{dx}(\sec x) = \sec x \tan x$

8. $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

9. $\frac{d}{dx}(\tan x) = \sec^2 x$

10. $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

2. Basic Theorems

1. $\frac{d}{dx}(f \pm g) = f'(x) \pm g'(x)$

2. $\frac{d}{dx}(k\, f(x)) = k \frac{d}{dx} f(x)$

3. $\frac{d}{dx}(f(x) \,.\, g(x)) = f(x)\, g'(x) + g(x)\, f'(x)$

4. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

5. $\frac{d}{dx}(f(g(x))) = f'(g(x))\, g'(x)$

Derivative Of Inverse Trigonometric Functions.

$$\frac{d\sin^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d\cos^{-1}x}{dx} = -\frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1.$$

$$\frac{d\tan^{-1}x}{dx} = \frac{1}{1+x^2}, \quad \frac{d\cot^{-1}x}{dx} = -\frac{1}{1+x^2} \quad (x \in R)$$

$$\frac{d\sec^{-1}x}{dx} = \frac{1}{|x|\sqrt{x^2-1}}, \quad \frac{d\,\mathrm{cosec}^{-1}x}{dx}$$

$$= -\frac{1}{|x|\sqrt{x^2-1}}, \text{ for } x \in (-\infty, -1) \cup (1, \infty)$$

3. Differentiation using substitution

Following substitutions are normally used to simplify these expression.

(i) $\sqrt{x^2+a^2}$ by substituting $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(ii) $\sqrt{a^2-x^2}$ by substituting $x = a\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

(iii) $\sqrt{x^2-a^2}$ by substituting $x = a\sec\theta$, where $\theta \in [0, \pi]$, $\theta \ne \frac{\pi}{2}$

(iv) $\sqrt{\frac{x+a}{a-x}}$ by substituting $x = a\cos\theta$, where $\theta \in (0, \pi]$.

4. **Parametric Differentiation**

If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter, then $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$.

5. **Derivative of one function with respect to another**

Let $y = f(x)$; $z = g(x)$ then $\dfrac{dy}{dz} = \dfrac{dy/dx}{dz/dx} = \dfrac{f'(x)}{g'(x)}$.

6. If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h, l, m, n, u, v, w are differentiable

functions of x then $F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} +$

$$\begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

APPLICATION OF DERIVATIVES

1. **Equation of tangent and normal**

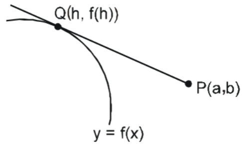
Tangent at (x_1, y_1) is given by $(y - y_1) = f'(x_1)(x - x_1)$; when, $f'(x_1)$ is real.

And normal at (x_1, y_1) is $(y - y_1) = -\dfrac{1}{f'(x_1)}(x - x_1)$, when $f'(x_1)$ is nonzero real.

2. **Tangent from an external point**

Given a point P(a, b) which does not lie on the curve y = f(x), then the equation of possible tangents to the curve y = f(x), passing through (a, b) can be found by solving for the point of contact Q.

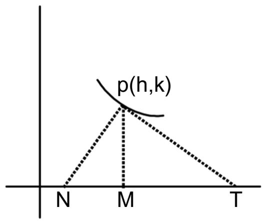
$$f'(h) = \frac{f(h) - b}{h - a}$$



Figure

And equation of tangent is $y - b = \frac{f(h)-b}{h-a}(x - a)$

3. Length of tangent, normal, subtangent, subnormal



(i) $PT = |k|\sqrt{1+\frac{1}{m^2}}$ = Length of Tangent

(ii) $PN = |k|\sqrt{1+m^2}$ = Length of Normal

(iii) $TM = \left|\frac{k}{m}\right|$ = Length of subtangent

(iv) $MN = |km|$ = Length of subnormal.

4. Angle between the curves

Angle between two intersecting curves is defined as the acute angle between their tangents (or normals) at the point of intersection of two curves (as shown in figure).

$$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$$

5. Shortest distance between two curves

Shortest distance between two non-intersecting differentiable curves is always along their common normal.
(Wherever defined)

6. Rolle's Theorem :

If a function f defined on [a, b] is

(i) continuous on [a, b]

(ii) derivable on (a, b) and

(iii) f(a) = f(b),

then there exists at least one real number c between a and b ($a < c < b$) such that $f'(c) = 0$

7. **Lagrange's Mean Value Theorem (LMVT) :**

If a function f defined on [a, b] is

(i) continuous on [a, b] and (ii) derivable on (a, b)

then there exists at least one real numbers between a and b (a < c < b) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$

8. **Useful Formulae of Mensuration to Remember :**

1. Volume of a cuboid = ℓbh.
2. Surface area of cuboid = $2(\ell b + bh + h\ell)$.
3. Volume of cube = a^3
4. Surface area of cube = $6a^2$
5. Volume of a cone = $\frac{1}{3}\pi r^2 h$.
6. Curved surface area of cone = $\pi r \ell$ (ℓ = slant height)
7. Curved surface area of a cylinder = $2\pi rh$.
8. Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.
9. Volume of a sphere = $\frac{4}{3}\pi r^3$.
10. Surface area of a sphere = $4\pi r^2$.
11. Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.
12. Volume of a prism = (area of the base) × (height).
13. Lateral surface area of a prism = (perimeter of the base) × (height).
14. Total surface area of a prism = (lateral surface area) + 2 (area of the base)
(Note that lateral surfaces of a prism are all rectangle).
15. Volume of a pyramid = $\frac{1}{3}$ (area of the base) × (height).
16. Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) × (slant height).
(Note that slant surfaces of a pyramid are triangles).

INDEFINITE INTEGRATION

1. If f & g are functions of x such that $g'(x) = f(x)$ then,

$\int f(x)\,dx = g(x) + c \Leftrightarrow \frac{d}{dx}\{g(x)+c\} = f(x)$, where c is called the **constant of integration**.

2. Standard Formula:

(i) $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,\ n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b) + c$

(iii) $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$

(iv) $\int a^{px+q}\,dx = \frac{1}{p}\frac{a^{px+q}}{\ln a} + c;\ a > 0$

(v) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$

(vi) $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

(vii) $\int \tan(ax+b)\,dx = \frac{1}{a}\ln\sec(ax+b) + c$

(viii) $\int \cot(ax+b)\,dx = \frac{1}{a}\ln\sin(ax+b) + c$

(ix) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$

(x) $\int \mathrm{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + c$

(xi) $\int \sec x\,dx = \ln(\sec x + \tan x) + c$ **OR** $\ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

(xii) $\int \operatorname{cosec} x\,dx = \ln(\operatorname{cosec} x - \cot x) + c$

OR $\ln \tan \frac{x}{2} + c$ **OR** $-\ln(\operatorname{cosec} x + \cot x) + c$

(xiii) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$

(x iv) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(xv) $\int \frac{dx}{|x|\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(xvi) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left[x + \sqrt{x^2 + a^2}\right] + c$

(xvii) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left[x + \sqrt{x^2 - a^2}\right] + c$

(xviii) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

(xix) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

(xx) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(xxi) $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right) + c$

(xxii) $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) + c$

3. Integration by Subsitutions

If we subsitute f(x) = t, then $f'(x)\,dx = dt$

4. Integration by Part :

$$\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int\left(\frac{d}{dx}f(x)\int g(x)\,dx\right)dx$$

5. Integration of type $\int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \int \sqrt{ax^2 + bx + c}\,dx$

Make the substitution $x + \frac{b}{2a} = t$

6. Integration of type

$$\int \frac{px+q}{ax^2+bx+c}\,dx, \int \frac{px+q}{\sqrt{ax^2 + bx + c}}\,dx, \int (px+q)\sqrt{ax^2 + bx + c}\,dx$$

Make the substitution x + $\frac{b}{2a}$ = t, then split the integral as some of two integrals one containing the linear term and the other containing constant term.

7. Integration of trigonometric functions

(i) $\int \frac{dx}{a + b\sin^2 x}$ **OR** $\int \frac{dx}{a + b\cos^2 x}$

OR $\int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$ **put tan x = t.**

(ii) $\int \frac{dx}{a + b\sin x}$ **OR** $\int \frac{dx}{a + b\cos x}$

OR $\int \frac{dx}{a + b\sin x + c\cos x}$ **put tan** $\frac{x}{2}$ **= t**

(i ii) $\int \frac{a\cos x + b\sin x + c}{\ell\cos x + m\sin x + n}$ **dx. Express Nr ≡ A(Dr) + B** $\frac{d}{dx}$**(Dr) + c & proceed.**

8. **Integration of type**

$$\int \frac{x^2 \pm 1}{x^4 + Kx^2 + 1}\, dx$$ where K is any constant.

Divide Nr & Dr by x² & put $x \mp \frac{1}{x}$ = t.

9. **Integration of type**

$$\int \frac{dx}{(ax+b)\sqrt{px+q}} \quad \textbf{OR} \quad \int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$$; put px + q = t^2.

10. **Integration of type**

$$\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$$, put ax + b = $\frac{1}{t}$;

$$\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$$, **put x =** $\frac{1}{t}$

DEFINITE INTEGRATION

Properties of definite integral

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

4. $\int_{-a}^{a} f(x)\,dx = \int_0^a (f(x) + f(-x))\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & f(-x) = f(x) \\ 0, & f(-x) = -f(x) \end{cases}$

5. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

FUNDAMENTAL OF MATHEMATICS

Intervals :

Intervals are basically subsets of R and are commonly used in solving inequalities or in finding domains. If there are two numbers a, b $\in$ R such that a < b, we can define four types of intervals as follows :

Symbols Used

(i) Open interval : (a, b) = {x : a < x < b} i.e. end points are not included. ( ) or] [

(ii) Closed interval : [a, b] = {x : a $\le$ x $\le$ b} i.e. end points are also included. [ ]

This is possible only when both a and b are finite.

(iii) Open-closed interval : (a, b] = {x : a < x $\le$ b} ( ] or]]

(iv) Closed - open interval : [a, b) = x : a $\le$ x < b} [ ) or [[

The infinite intervals are defined as follows :

(i) (a, ∞) = {x : x > a}

(ii) [a, ∞) = {x : x $\ge$ a}

(iii) (– ∞, b) = {x : x < b}

(iv) ($-\infty$, b] = {x : x $\le$ b}

(v) (– ∞, ∞) = {x : x $\in$ R}

Properties of Modulus :

For any a, b $\in$ R

|a| $\ge$ 0, |a| = |–a|, |a| $\ge$ a, |a| $\ge$ –a, |ab| = |a| |b|,

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, |a + b| $\le$ |a| + |b|, |a – b| $\ge$ ||a| – |b||

Trigonometric Functions of Sum or Difference of Two Angles:

(a) sin (A ± B) = sinA cosB ± cosA sinB

$\therefore$ 2 sinA cosB = sin(A+B) + sin(A–B) and

and 2 cosA sinB = sin(A+B) – sin(A–B)

(b) cos (A ± B) = cosA cosB $\mp$ sinA sinB

$\therefore$ 2 cosA cosB = cos(A+B) + cos(A–B) and 2sinA sinB = cos(A–B) – cos(A+B)

(c) $\sin^2A - \sin^2B = \cos^2B - \cos^2A = \sin(A+B).\sin(A-B)$

(d) $\cos^2A - \sin^2B = \cos^2B - \sin^2A = \cos(A+B).\cos(A-B)$

(e) $\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

(f) $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$.

3. Common Roots:

Consider two quadratic equations $a_1x^2 + b_1x + c_1 = 0$ & $a_2x^2 + b_2x + c_2 = 0$.

(i) If two quadratic equations have both roots common, then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

(ii) If only one root α is common, then

$$\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} = \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$$

4. Range of Quadratic Expression f (x) = $ax^2 + bx + c$.

Range in restricted domain: Given $x \in [x_1, x_2]$

(a) If $-\frac{b}{2a} \notin [x_1, x_2]$ then,

$$f(x) \in \left[\min\{f(x_1), f(x_2)\}, \max\{f(x_1), f(x_2)\}\right]$$

(b) If $-\frac{b}{2a} \in [x_1, x_2]$ then,

$$f(x) \in \left[\min\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}, \max\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}\right]$$

5. **Location of Roots:**

Let $f(x) = ax^2 + bx + c$, where $a > 0$ & $a, b, c \in R$.

(i) Conditions for both the roots of f (x) = 0 to be greater than a specified number 'x_0' are $b^2 - 4ac \geq 0$; $f(x_0) > 0$ & $(-b/2a) > x_0$.

(ii) Conditions for both the roots of f (x) = 0 to be smaller than a specified number 'x_0' are $b^2 - 4ac \geq 0$; $f(x_0) > 0$ & $(-b/2a) < x_0$.

(iii) Conditions for both roots of f (x) = 0 to lie on either side of the number 'x_0' (in other words the number 'x_0' lies between the roots of f (x) = 0), is $f(x_0) < 0$.

(iv) Conditions that both roots of f (x) = 0 to be confined between the numbers x_1 and x_2, $(x_1 < x_2)$ are $b^2 - 4ac \geq 0$; $f(x_1) > 0$; $f(x_2) > 0$ & $x_1 < (-b/2a) < x_2$.

(v) Conditions for exactly one root of f (x) = 0 to lie in the interval (x_1, x_2) i.e. $x_1 < x < x_2$ is $f(x_1).f(x_2) < 0$.

SEQUENCE & SERIES

An arithmetic progression (A.P.) : a, a + d, a + 2 d,..... a + (n − 1) d is an A.P.

Let a be the first term and d be the common difference of an A.P., then nth term = $t_n = a + (n - 1) d$

The sum of first n terms of A.P. are

$S_n = \frac{n}{2}[2a + (n - 1) d] = \frac{n}{2}[a + □□]$

rth term of an A.P. when sum of first r terms is given is $t_r = S_r - S_{r-1}$.

Properties of A.P.

(i) If a, b, c are in A.P. ⇔ 2 b = a + c & if a, b, c, d are in A.P. ⇒ a + d = b + c.

(ii) Three numbers in A.P. can be taken as a − d, a, a + d; four numbers in A.P. can be taken as a − 3d, a − d, a + d, a + 3d; five numbers in A.P. are a − 2d, a − d, a, a + d, a + 2d & six terms in A.P. are a − 5d, a − 3d, a − d, a + d, a + 3d, a + 5d etc.

(iii) Sum of the terms of an A.P. equidistant from the beginning & end = sum of first & last term.

Arithmetic Mean (Mean or Average) (A.M.):

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a & c.

n – Arithmetic Means Between Two Numbers:
If a, b are any two given numbers & a, $A_1, A_2, \ldots, A_n$, b are in A.P. then $A_1, A_2, \ldots A_n$ are the n A.M.'s between a & b. $A_1 = a + \frac{b - a}{n + 1}$,

$A_2 = a + \frac{2(b - a)}{n + 1}$,......, $A_n = a + \frac{n(b-a)}{n + 1}$

$\sum_{r=1}^{n} A_r = nA$ where A is the single A.M. between a & b.

Geometric Progression:

a, ar, ar^2, ar^3, ar^4,...... is a G.P. with a as the first term & r as common ratio.

(i) nth term = $a r^{n-1}$

(ii) Sum of the first n terms i.e. $S_n = \begin{cases} \frac{a(r^n - 1)}{r - 1}, & r \neq 1 \\ na, & r = 1 \end{cases}$

(iii) Sum of an infinite G.P. when $|r| < 1$ is given by

$S_\infty = \frac{a}{1 - r} \quad (|r| < 1)$.

Geometric Means (Mean Proportional) (G.M.):

If a, b, c > 0 are in G.P., b is the G.M. between a & c, then $b^2 = ac$

n–Geometric Means Between positive number a, b: If a, b are two given numbers & a, $G_1, G_2, \ldots, G_n$, b are in G.P.. Then $G_1, G_2, G_3, \ldots, G_n$ are nG.M.s between a & b.

$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$,......, $G_n = a(b/a)^{n/n+1}$

Harmonic Mean (H.M.):

If a, b, c are in H.P., b is the H.M. between a & c, then $b = \frac{2ac}{a + c}$.

H.M. H of a_1, a_2, ... a_n is given by $\frac{1}{H} = \frac{1}{n}\left[\frac{1}{a_1} + \frac{1}{a_2} + \ldots\ldots + \frac{1}{a_n}\right]$

Relation between means :

$G^2 = AH$, A.M. $\geq$ G.M. $\geq$ H.M. (only for two numbers)

and A.M. = G.M. = H.M. if $a_1 = a_2 = a_3 = \ldots\ldots\ldots = a_n$

Important Results

(i) $\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$. (ii) $\sum_{r=1}^{n} k\,a_r = k \sum_{r=1}^{n} a_r$.

(iii) $\sum_{r=1}^{n} k = nk$; where k is a constant.

(iv) $\sum_{r=1}^{n} r = 1 + 2 + 3 + \ldots\ldots\ldots + n = \frac{n(n+1)}{2}$

(v) $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(vi) $\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots\ldots + n^3 = \frac{n^2 (n+1)^2}{4}$

BINOMIAL THEOREM

1. Statement of Binomial theorem : If a, b $\in$ R and n $\in$ N, then

$(a + b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + {}^nC_n\, a^0 b^n$

$= \sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r$

2. Properties of Binomial Theorem :

(i) **General term :** $T_{r+1} = {}^nC_r\, a^{n-r} b^r$

(ii) **Middle term (s) :**

(a) If n is even, there is only one middle term,

which is $\left(\frac{n+2}{2}\right)$th term.

(b) If n is odd, there are two middle terms,

which are $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+1}{2}+1\right)$th terms.

3. Multinomial Theorem :

$(x_1 + x_2 + x_3 + \ldots\ldots\ldots\ldots x_k)^n$

$$= \sum_{r_1 + r_2 + \ldots + r_k = n} \frac{n!}{r_1!\, r_2! \ldots r_k!}\, x_1^{r_1} \cdot x_2^{r_2} \ldots x_k^{r_k}$$

Here total number of terms in the expansion = $^{n+k-1}C_{k-1}$

4. Application of Binomial Theorem :

If $(\sqrt{A} + B)^n = I + f$ where I and n are positive integers, n being odd and $0 < f < 1$ then $(I + f)\, f = k^n$ where $A - B^2 = k > 0$ and $\sqrt{A} - B < 1$.

If n is an even integer, then $(I + f)(1 - f) = k^n$

5. Properties of Binomial Coefficients :

(i) $^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots\ldots + {}^nC_n = 2^n$

(ii) $^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots\ldots\ldots\ldots + (-1)^n\, {}^nC_n = 0$

(iii) $^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = 2^{n-1}$

(iv) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ (v) $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n - r + 1}{r}$

6. Binomial Theorem For Negative Integer Or Fractional Indices

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots\ldots\ldots(n-r+1)}{r!}x^r + \ldots, |x| < 1.$$

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots\ldots\ldots(n-r+1)}{r!}x^r$$

PERMUTATION & COMBINNATION

1. Arrangement : number of permutations of n different things taken r at a time = $^nP_r = n(n - 1)(n - 2)\ldots(n - r + 1) = \dfrac{n!}{(n-r)!}$

2. Circular Permutation :
The number of circular permutations of n different things taken all at a time is; $(n - 1)!$

3. Selection : Number of combinations of n different things taken r at a time = $^nC_r = \dfrac{n!}{r!(n-r)!} = \dfrac{^nP_r}{r!}$

4. The number of permutations of 'n' things, taken all at a time, when 'p' of them are similar & of one type, q of them are similar & of another type, 'r' of them are similar & of a third type & the remaining $n - (p + q + r)$ are all different is $\dfrac{n!}{p!\,q!\,r!}$.

5. Selection of one or more objects

(a) Number of ways in which atleast one object be selected out of 'n' distinct objects is

$$^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = 2^n - 1$$

(b) Number of ways in which atleast one object may be selected out of 'p' alike objects of one type 'q' alike objects of second type and 'r' alike of third type is

$$(p + 1)(q + 1)(r + 1) - 1$$

(c) Number of ways in which atleast one object may be selected from 'n' objects where 'p' alike of one type 'q' alike of second type and 'r' alike of third type and rest $n - (p + q + r)$ are different, is

$$(p + 1)(q + 1)(r + 1)\,2^{n - (p + q + r)} - 1$$

6. Multinomial Theorem :
Coefficient of x^r in expansion of $(1 - x)^{-n} = {}^{n+r-1}C_r \ (n \in N)$

7. Let $N = p^a.\ q^b.\ r^c\ldots$ where p, q, r...... are distinct primes & a, b, c..... are natural numbers then :

(a) The total numbers of divisors of N including 1 & N is

$$= (a + 1)(b + 1)(c + 1)\ldots$$

(b) The sum of these divisors is =

$$(p^0 + p^1 + p^2 + \ldots + p^a)(q^0 + q^1 + q^2 + \ldots + q^b)(r^0 + r^1 + r^2 + \ldots + r^c)\ldots$$

(c) Number of ways in which N can be resolved as a product of two factors is

$$= \begin{cases} \frac{1}{2}(a+1)(b+1)(c+1).... & \text{if N is not a perfect square} \\ \frac{1}{2}[(a+1)(b+1)(c+1)....+1] & \text{if N is a perfect square} \end{cases}$$

(d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

8. Dearrangement :

Number of ways in which 'n' letters can be put in 'n' corresponding envelopes such that no letter goes to correct envelope is

$$n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \text{..........} + (-1)^n \frac{1}{n!}\right]$$

PROBABILITY

1. Classical (A priori) Definition of Probability :

If an experiment results in a total of (m + n) outcomes which are equally likely and mutually exclusive with one another and if 'm' outcomes are favorable to an event 'A' while 'n' are unfavorable, then the probability of occurrence of the event 'A' = $P(A) = \frac{m}{m+n} = \frac{n(A)}{n(S)}$.

We say that odds in favour of 'A' are m : n, while odds against 'A' are n : m.

$$P(\bar{A}) = \frac{n}{m+n} = 1 - P(A)$$

2. Addition theorem of probability : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

De Morgan's Laws :

(a) $(A \cup B)^c = A^c \cap B^c$ (b) $(A \cap B)^c = A^c \cup B^c$

Distributive Laws :

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(i) $P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

(ii) P (at least two of A, B, C occur) = $P(B \cap C) + P(C \cap A) + P(A \cap B) - 2P(A \cap B \cap C)$

(iii) P(exactly two of A, B, C occur) = $P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$

(iv) P(exactly one of A, B, C occur) =

$P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C)$

3. **Conditional Probability :** $P(A/B) = \frac{P(A \cap B)}{P(B)}$.

4. **Binomial Probability Theorem**

If an experiment is such that the probability of success or failure does not change with trials, then the probability of getting exactly r success in n trials of an experiment is $nC_r \, p^r q^{n-r}$, where 'p' is the probability of a success and q is the probability of a failure. Note that p + q = 1.

5. **Expectation :**

If a value Mi is associated with a probability of pi , then the expectation is given by $\sum p_i M_i$.

6. **Total Probability Theorem :** $P(A) = \sum_{i=1}^{n} P(Bi).P(A/Bi)$

7. **Bayes' Theorem :**

If an event A can occur with one of the n mutually exclusive and exhaustive events B1, B2 , ..., Bn and the probabilities P(A/B1), P(A/B2) .... P(A/Bn) are known, then $P(Bi / A) = \frac{P(Bi).P(A/Bi)}{\sum_{i=1}^{n} P(Bi).P(A/Bi)}$ B1, B2, B3,........,Bn

$A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup \ldots \cup (A \cap B_n)$

$P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots + P(A \cap B_n) = \sum_{i=1}^{n} P(A \cap B_i)$

8. **Binomial Probability Distribution :**

(i) Mean of any probability distribution of a random variable is given by :

$\mu = \frac{\sum p_i x_i}{\sum p_i} = \sum p_i x_i = np$

n = number of trials

p = probability of success in each probability

q = probability of failure

(ii) Variance of a random variable is given by,

$\sigma^2 = \sum (x_i - \mu)^2 . p_i = \sum p_i x_i^2 - \mu^2 = npq$

COMPLEX NUMBER

1. The complex number system

$z = a + ib$, then $a - ib$ is called congugate of z and is denoted by $\bar{z}$.

2. Equality In Complex Number:

$$z_1 = z_2 \Leftrightarrow Re(z_1) = Re(z_2) \text{ and } I_m(z_1) = I_m(z_2).$$

3. Properties of arguments

(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2m\pi$ for some integer m.

(ii) $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2) + 2m\pi$ for some integer m.

(iii) $\arg(z^2) = 2\arg(z) + 2m\pi$ for some integer m.

(iv) $\arg(z) = 0 \Leftrightarrow$ z is a positive real number

(v) $\arg(z) = \pm \pi/2 \Leftrightarrow$ z is purely imaginary and $z \neq 0$

4. Properties of conjugate

(i) $|z| = |\bar{z}|$ (ii) $z\bar{z} = |z|^2$ (iii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(iv) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ (v) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(vi) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ $(z_2 \neq 0)$

(vii) $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2$

(viii) $\overline{(\bar{z}_1)} = z$ (ix) If $w = f(z)$, then $\bar{w} = f(\bar{z})$

(x) $\arg(z) + \arg(\bar{z})$

5. Rotation theorem

If $P(z_1)$, $Q(z_2)$ and $R(z_3)$ are three complex numbers and $\angle PQR = \theta$, then

$$\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \left|\frac{z_3 - z_2}{z_1 - z_2}\right| e^{i\theta}$$

6. Demoivre's Theorem :

If n is any integer then

(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ldots(\cos\theta_n + i\sin\theta_n) = \cos(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n)$

7. Cube Root Of Unity :

(i) The cube roots of unity are 1, $\frac{-1+i\sqrt{3}}{2}$, $\frac{-1-i\sqrt{3}}{2}$.

(ii) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$.
In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3.

8. Geometrical Properties:

Distance formula : $|z_1 - z_2|$.

Section formula : $z = \frac{mz_2 + nz_1}{m + n}$ (internal division), $z = \frac{mz_2 - nz_1}{m - n}$ (external division)

(1) amp(z) = θ is a ray emanating from the origin inclined at an angle θ to the x− axis.

(2) $|z - a| = |z - b|$ is the perpendicular bisector of the line joining a to b.

(3) If $\left|\frac{z - z_1}{z - z_2}\right| = k \neq 1, 0$, then locus of z is circle.

VECTORS

- **Position Vector Of A Point**:
let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B, then,
$\overrightarrow{AB} = \vec{b} - \vec{a}$ = pv of B − pv of A.

DISTANCE FORMULA : Distance between the two points $A(\vec{a})$ and $B(\vec{b})$ is $AB = |\vec{a} - \vec{b}|$

SECTION FORMULA : $\vec{r} = \frac{n\vec{a} + m\vec{b}}{m + n}$. Mid point of AB = $\frac{\vec{a} + \vec{b}}{2}$.

II. **Scalar Product Of Two Vectors:** $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$, where $|\vec{a}|, |\vec{b}|$ are magnitude of $\vec{a}$ and $\vec{b}$ respectively and θ is angle between $\vec{a}$ and $\vec{b}$.

1. $i.i = j.j = k.k = 1$; $i.j = j.k = k.i = 0$ $\Rightarrow$ projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

2. If $\vec{a} = a_1 i + a_2 j + a_3 k$ & $\vec{b} = b_1 i + b_2 j + b_3 k$ then $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

3. The angle θ between $\vec{a}$ & $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$, $0 \le \theta \le \pi$

4. $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ $\quad(\vec{a} \neq 0,\ \vec{b} \neq 0)$

III. **Vector Product Of Two Vectors:**

1. If $\vec{a}$ & $\vec{b}$ are two vectors & θ is the angle between them then $\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}, \vec{b}$ & $\hat{n}$ forms a right handed screw system.

2. Geometrically $|\vec{a}\times\vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ & $\vec{b}$.

3. $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$; $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$

4. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

5. $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel (collinear) $(\vec{a} \neq 0, \vec{b} \neq 0)$ i.e. $\vec{a} = K\vec{b}$, where K is a scalar.

6. Unit vector perpendicular to the plane of $\vec a$ & $\vec b$ is $\hat n = \pm \frac{\vec a \times \vec b}{|\vec a \times \vec b|}$

- If $\vec a, \vec b$ & $\vec c$ are the pv's of 3 points A, B & C then the vector area of triangle ABC = $\frac{1}{2}[\vec a \times \vec b + \vec b \times \vec c + \vec c \times \vec a]$.

  The points A, B & C are collinear if $\vec a \times \vec b + \vec b \times \vec c + \vec c \times \vec a = \vec 0$

- Area of any quadrilateral whose diagonal vectors are $\vec d_1$ & $\vec d_2$ is given by $\frac{1}{2}|\vec d_1 \times \vec d_2|$

- Lagrange's Identity : $(\vec a \times \vec b)^2 = |\vec a|^2 |\vec b|^2 - (\vec a \cdot \vec b)^2 = \begin{vmatrix} \vec a \cdot \vec a & \vec a \cdot \vec b \\ \vec a \cdot \vec b & \vec b \cdot \vec b \end{vmatrix}$

IV. Scalar Triple Product:

- The scalar triple product of three vectors $\vec a, \vec b$ & $\vec c$ is defined as:

  $\vec a \times \vec b \cdot \vec c = |\vec a||\vec b||\vec c| \sin\theta \cos\phi$.

- Volume of tetrahydron $V = [\vec a \vec b \vec c]$
- In a scalar triple product the position of dot & cross can be interchanged i.e.

  $\vec a \cdot (\vec b \times \vec c) = (\vec a \times \vec b) \cdot \vec c$ OR $[\vec a\ \vec b\ \vec c] = [\vec b\ \vec c\ \vec a] = [\vec c\ \vec a\ \vec b]$

- $\vec a \cdot (\vec b \times \vec c) = -\vec a \cdot (\vec c \times \vec b)$ i.e. $[\vec a\ \vec b\ \vec c] = -[\vec a\ \vec c\ \vec b]$

- If $\vec a = a_1 i + a_2 j + a_3 k$; $\vec b = b_1 i + b_2 j + b_3 k$ & $\vec c = c_1 i + c_2 j + c_3 k$ then $[\vec a \vec b \vec c] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

- If $\vec a, \vec b, \vec c$ are coplanar $\Leftrightarrow [\vec a \vec b \vec c] = 0$.

- Volume of tetrahedron OABC with O as origin & A($\vec a$), B($\vec b$) and C($\vec c$) be the vertices = $\left|\frac{1}{6}[\vec a\ \vec b\ \vec c]\right|$

- The positon vector of the centroid of a tetrahedron if the pv's of its vertices are $\vec{a},\vec{b},\vec{c}$ & $\vec{d}$ are given by $\frac{1}{4}[\vec{a}+\vec{b}+\vec{c}+\vec{d}]$.

V. **Vector Triple Product:**

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$, $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

- $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$, in general

3-DIMENSION

1. Vector representation of a point :

Position vector of point P (x, y, z) is $x\hat{i} + y\hat{j} + z\hat{k}$.

2. Distance formula :

$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$, AB = $|\overrightarrow{OB} - \overrightarrow{OA}|$

3. Distance of P from coordinate axes :

PA = $\sqrt{y^2+z^2}$, PB = $\sqrt{z^2+x^2}$, PC = $\sqrt{x^2+y^2}$

4. Section Formula : $x = \frac{mx_2+nx_1}{m+n}$, $y = \frac{my_2+ny_1}{m+n}$, $z = \frac{mz_2+nz_1}{m+n}$

Mid point : $x = \frac{x_1+x_2}{2}, y = \frac{y_1+y_2}{2}, z = \frac{z_1+z_2}{2}$

5. Direction Cosines And Direction Ratios

(i) Direction cosines: Let α, β, γ be the angles which a directed line makes with the positive directions of the axes of x, y and z respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines of the line. The direction cosines are usually denoted by (ℓ, m, n). Thus $\ell = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

(ii) If ℓ, m, n be the direction cosines of a line, then $\ell^2 + m^2 + n^2 = 1$

(iii) Direction ratios: Let a, b, c be proportional to the direction cosines ℓ, m, n then a, b, c are called the direction ratios.

(iv) If ℓ, m, n be the direction cosines and a, b, c be the direction ratios of a vector, then

$$\ell = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

(vi) If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2) then the direction ratios of line PQ are, $a = x_2 - x_1$, $b = y_2 - y_1$ & $c = z_2 - z_1$ and the direction cosines of line PQ are $\ell = \frac{x_2 - x_1}{|PQ|}$,

$$m = \frac{y_2 - y_1}{|PQ|} \text{ and } n = \frac{z_2 - z_1}{|PQ|}$$

6. Angle Between Two Line Segments:

$$\cos\theta = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|.$$

The line will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$,

parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

7. Projection of a line segment on a line

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ then the projection of PQ on a line having direction cosines ℓ, m, n is

$$\left| \ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1) \right|$$

8. Equation Of A Plane :

General form: $ax + by + cz + d = 0$, where a, b, c are not all zero, $a, b, c, d \in R$.

(i) Normal form : $\ell x + my + nz = p$

(ii) Plane through the point (x_1, y_1, z_1) :

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

(iii) Intercept Form: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

(iv) Vector form: $(\vec{r} - \vec{a}).\vec{n} = 0$ or $\vec{r}.\vec{n} = \vec{a}.\vec{n}$

(v) Any plane parallel to the given plane $ax + by + cz + d = 0$ is $ax + by + cz + \lambda = 0$. Distance between $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $= \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$

(vi) Equation of a plane passing through a given point & parallel to the given vectors:

$\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ (parametric form) where λ & μ are scalars.

or $\vec{r}.(\vec{b} \times \vec{c}) = \vec{a}.(\vec{b} \times \vec{c})$ (non parametric form)

9. A Plane & A Point

(i) Distance of the point (x', y', z') from the plane $ax + by + cz + d = 0$ is given by $\dfrac{ax' + by' + cz' + d}{\sqrt{a^2 + b^2 + c^2}}$.

(ii) Length of the perpendicular from a point ($\vec{a}$) to plane $\vec{r}.\vec{n} = d$ is given by $p = \dfrac{|\vec{a}.\vec{n} - d|}{|\vec{n}|}$.

(iii) Foot (x', y', z') of perpendicular drawn from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is given by $\dfrac{x' - x_1}{a} = \dfrac{y' - y_1}{b} = \dfrac{z' - z_1}{c}$

$= -\dfrac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$

(iv) To find image of a point w.r.t. a plane:

Let P (x_1, y_1, z_1) is a given point and $ax + by + cz + d = 0$ is given plane Let (x', y', z') is the image point. then

$$\frac{x' - x_1}{a} = \frac{y' - y_1}{b} = \frac{z' - z_1}{c} = -2\frac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

10. Angle Between Two Planes:

$$\cos\theta = \left|\frac{aa' + bb' + cc'}{\sqrt{a^2 + b^2 + c^2}\sqrt{a'^2 + b'^2 + c'^2}}\right|$$

Planes are perpendicular if $aa' + bb' + cc' = 0$ and planes are parallel if

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

The angle θ between the planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is given by,

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \, |\vec{n}_2|}$$

Planes are perpendicular if $\vec{n}_1 \cdot \vec{n}_2 = 0$ & planes are parallel if $\vec{n}_1 = \lambda \vec{n}_2$, λ is a scalar

11. Angle Bisectors

(i) The equations of the planes bisecting the angle between two given planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(ii) Bisector of acute/obtuse angle: First make both the constant terms positive. Then

$a_1a_2 + b_1b_2 + c_1c_2 > 0 \Rightarrow$ origin lies on obtuse angle

$a_1a_2 + b_1b_2 + c_1c_2 < 0 \Rightarrow$ origin lies in acute angle

12. Family of Planes

(i) Any plane through the intersection of $a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ is

$$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0$$

(ii) The equation of plane passing through the intersection of the planes $\vec{r} \cdot \vec{n}_1 = d_1$ & $\vec{r} \cdot \vec{n}_2 = d_2$ is $\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + \lambda d_2$ where λ is arbitrary scalar

13. Volume Of A Tetrahedron

Volume Of A Tetrahedron: Volume of a tetrahedron with vertices A (x_1, y_1, z_1), B(x_2, y_2, z_2), C (x_3, y_3, z_3) and D (x_4, y_4, z_4) is given by

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

A LINE

1. Equation Of A Line

(i) A straight line is intersection of two planes.

it is reprsented by two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$.

(ii) Symmetric form : $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r$.

(iii) Vector equation: $\vec{r} = \vec{a} + \lambda\vec{b}$

(iv) Reduction of cartesion form of equation of a line to vector form & vice versa

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \Leftrightarrow \vec{r} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k}).$$

2. Angle Between A Plane And A Line:

(i) If θ is the angle between line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and the plane $ax + by + cz + d = 0$, then

$$\sin\theta = \left| \frac{a\ell + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{\ell^2 + m^2 + n^2}} \right|.$$

(ii) Vector form: If θ is the angle between a line $\vec{r} = (\vec{a} + \lambda\vec{b})$ and $\vec{r} \cdot \vec{n} = d$ then $\sin\theta = \left(\frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|} \right)$.

(iii) Condition for perpendicularity $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$, $\vec{b} \times \vec{n} = 0$

(iv) Condition for parallel $a\ell + bm + cn = 0$ $\vec{b} \cdot \vec{n} = 0$

3. Condition For A Line To Lie In A Plane

(i) Cartesian form: Line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ would lie in a plane $ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d = 0$ & $a\ell + bm + cn = 0$.

(ii) Vector form: Line $\vec{r} = \vec{a} + \lambda\vec{b}$ would lie in the plane $\vec{r} \cdot \vec{n} = d$ if $\vec{b} \cdot \vec{n} = 0$ & $\vec{a} \cdot \vec{n} = d$

4. Skew Lines:

(i) The straight lines which are not parallel and non-coplanar i.e. non-intersecting are called skew lines.

lines $\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ & $\frac{x-\alpha'}{\ell'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$

If $\Delta = \begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \neq 0$, then lines are skew.

(ii) Shortest distance formula for lines

$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ is $d = \left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$

(iii) Vector Form: For lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ to be skew

$(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1) \neq 0$

(iv) Shortest distance between parallel lines

$\vec{r} = \vec{a}_1 + \lambda\vec{b}$ & $\vec{r} = \vec{a}_2 + \mu\vec{b}$ is $d = \left|\frac{(\vec{a}_2-\vec{a}_1)\times\vec{b}}{|\vec{b}|}\right|$.

(v) Condition of coplanarity of two lines $\vec{r} = \vec{a} + \lambda\vec{b}$ & $\vec{r} = \vec{c} + \mu\vec{d}$ is

$[\vec{a}-\vec{c}\ \ \vec{b}\ \ \vec{d}] = 0$

5. Sphere

General equation of a sphere is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

$(-u, -v, -w)$ is the centre and $\sqrt{u^2+v^2+w^2-d}$ is the radius of the sphere.

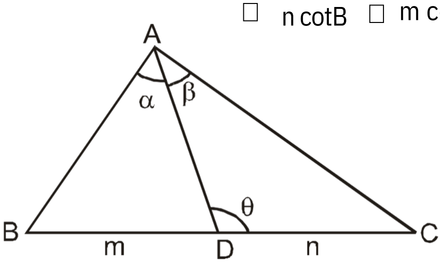
6. Area of Triangle (Δ)

$$\Delta = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \sqrt{s(s-a)(s-b)(s-c)}$$

7. m - n Rule:

If BD : DC = m : n, then

$$(m + n) \cot \theta = m \cot \alpha - n \cot \beta$$
$$= n \cot B - m \cot C$$



8. Radius of Circumcirlce :

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} = \frac{abc}{4\Delta}$$

9. Radius of The Incircle :

(i) $r = \frac{\Delta}{s}$

(ii) $r = (s-a) \tan \frac{A}{2} = (s-b) \tan \frac{B}{2} = (s-c) \tan \frac{C}{2}$

(iii) $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

10. Radius of The Ex- Circles :

(i) $r1 = \frac{\Delta}{s-a}$; $r2 = \frac{\Delta}{s-b}$; $r^3 = \frac{\Delta}{s-c}$

(ii) $r1 = s \tan \frac{A}{2}$; $r2 = s \tan \frac{B}{2}$; $r3 = s \tan \frac{C}{2}$

(iii) $r1 = 4 R \sin \frac{A}{2} . \cos \frac{B}{2} . \cos \frac{C}{2}$

SOLUTION OF TRIANGLE

1. Sine Rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

2. Cosine Formula:

(i) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ (ii) $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$

(iii) $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

3. Projection Formula:

(i) $a = b \cos C + c \cos B$ (ii) $b = c \cos A + a \cos C$ (iii) $c = a \cos B + b \cos A$

4. Napier's Analogy - tangent rule:

(i) $\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}$ (ii) $\tan \frac{C - A}{2} = \frac{c - a}{c + a} \cot \frac{B}{2}$

(iii) $\tan \frac{A - B}{2} = \frac{a - b}{a + b} \cot \frac{C}{2}$

5. Trigonometric Functions of Half Angles:

(i) $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$; $\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$;

$\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

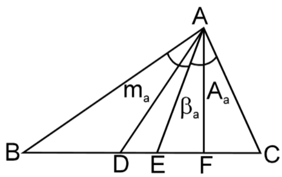
(ii) $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$; $\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$; $\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(iii) $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$ where $s = \frac{a+b+c}{2}$ is semi perimetre of triangle.

(iv) $\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$

11. Length of Angle Bisectors, Medians & Altitudes :

(i) Length of an angle bisector from the angle A = $\beta_a = \dfrac{2bc\cos\frac{A}{2}}{b+c}$;

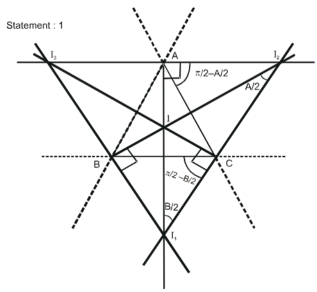


(ii) Length of median from the angle A = ma = $\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$

& (iii) Length of altitude from the angle A = A a = $\dfrac{2\Delta}{a}$

12. Orthocentre and Pedal Triangle:

The triangle KLM which is formed by joining the feet of the altitudes is called the Pedal Triangle.



(i) Its angles are $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$.

(ii) Its sides are a cosA = R sin 2A,

b cosB = R sin 2B and

c cosC = R sin 2C

(iii) Circumradii of the triangles PBC, PCA, PAB and ABC are equal.

13. Excentral Triangle:

The triangle formed by joining the three excentres I_1, I_2 and I_3 of Δ ABC is called the excentral or excentric triangle.

(i) Δ ABC is the pedal triangle of the $\Delta I_1 I_2 I_3$.

(ii) Its angles are $\frac{\pi}{2} - \frac{A}{2}$, $\frac{\pi}{2} - \frac{B}{2}$ & $\frac{\pi}{2} - \frac{C}{2}$.

(iii) Its sides are $4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$ & $4R\cos\frac{C}{2}$.

(iv) $II_1 = 4R\sin\frac{A}{2}$; $II_2 = 4R\sin\frac{B}{2}$; $II_3 = 4R\sin\frac{C}{2}$.

(v) Incentre I of Δ ABC is the orthocentre of the excentral $\Delta I_1 I_2 I_3$.

14. Distance Between Special Points :

(i) Distance between circumcentre and orthocentre

$$OH^2 = R^2(1 - 8\cos A\cos B\cos C)$$

(ii) Distance between circumcentre and incentre

$$OI^2 = R^2\left(1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = R^2 - 2Rr$$

(iii) Distance between circumcentre and centroid

$$OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$$

INVERSE TRIGONOMETRIC FUNCTIONS

1. Principal Values & Domains of Inverse Trigonometric/Circular Functions:

	Function	Domain	Range
(i)	$y = \sin^{-1} x$ where	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
(ii)	$y = \cos^{-1} x$ where	$-1 \le x \le 1$	$0 \le y \le \pi$
(iii)	$y = \tan^{-1} x$ where	$x \in R$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
(iv)	$y = \operatorname{cosec}^{-1} x$ where	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$
(v)	$y = \sec^{-1} x$ where	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi$; $y \ne \frac{\pi}{2}$
(vi)	$y = \cot^{-1} x$ where	$x \in R$	$0 < y < \pi$

P - 2 (i) $\sin^{-1}(\sin x) = x,\quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(ii) $\cos^{-1}(\cos x) = x;\quad 0 \le x \le \pi$

(iii) $\tan^{-1}(\tan x) = x;\quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

(iv) $\cot^{-1}(\cot x) = x;\quad 0 < x < \pi$

(v) $\sec^{-1}(\sec x) = x;\quad 0 \le x \le \pi,\ x \ne \frac{\pi}{2}$

(vi) $\text{cosec}^{-1}(\text{cosec}\, x) = x;\quad x \ne 0,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

P - 3 (i) $\sin^{-1}(-x) = -\sin^{-1}x,\quad -1 \le x \le 1$

(ii) $\tan^{-1}(-x) = -\tan^{-1}x,\quad x \in R$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1}x,\quad -1 \le x \le 1$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x,\quad x \in R$

P - 5 (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2},\quad -1 \le x \le 1$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2},\quad x \in R$

(iii) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2},\quad |x| \ge 1$

2. Identities of Addition and Substraction:

I - 1 (i) $\sin^{-1}x + \sin^{-1}y$

$= \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right],\quad x \ge 0,\ y \ge 0\ \&\ (x^2 + y^2) \le 1$

$= \pi - \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right],\quad x \ge 0,\ y \ge 0\ \&\ x^2 + y^2 > 1$

(ii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right],\quad x \ge 0,\ y \ge 0$

(iii) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy},\quad x > 0,\ y > 0\ \&\ xy < 1$

$= \pi + \tan^{-1}\frac{x+y}{1-xy},\quad x > 0,\ y > 0\ \&\ xy > 1$

$= \frac{\pi}{2},\quad x > 0,\ y > 0\ \&\ xy = 1$

I - 2 (i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$, $x \geq 0$, $y \geq 0$

(ii) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left[xy + \sqrt{1-x^2}\sqrt{1-y^2}\right]$, $x \geq 0$, $y \geq 0$, $x \leq y$

(iii) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$, $x > 0$, $y > 0$

I - 3 (i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{cases} 2\sin^{-1}x & \text{if } |x| \leq \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & \text{if } x > \frac{1}{\sqrt{2}} \\ -(\pi + 2\sin^{-1}x) & \text{if } x < -\frac{1}{\sqrt{2}} \end{cases}$

(ii) $\cos^{-1}(2x^2 - 1) = \begin{cases} 2\cos^{-1}x & \text{if } 0 \leq x \leq 1 \\ 2\pi - 2\cos^{-1}x & \text{if } -1 \leq x \leq 0 \end{cases}$

(iii) $\tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -(\pi - 2\tan^{-1}x) & \text{if } x > 1 \end{cases}$

(iv) $\sin^{-1}\dfrac{2x}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| \leq 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -(\pi + 2\tan^{-1}x) & \text{if } x < -1 \end{cases}$

(v) $\cos^{-1}\dfrac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } x \geq 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$

If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\dfrac{x + y + z - xyz}{1 - xy - yz - zx}\right]$ if, $x > 0$, $y > 0$, $z > 0$ & $(xy + yz + zx) < 1$

NOTE:

(i) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$ then $x + y + z = xyz$

(ii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \dfrac{\pi}{2}$ then $xy + yz + zx = 1$

(iii) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

STATISTICS

1. Arithmetic Mean / or Mean

If x1, x2, x3 ,.......xn are n values of variate xi then their A.M. $\bar{x}$ is defined as

$$\bar{x} = \frac{x_1 + x_2 + x_3 + ... + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

If x1, x2, x3, ... xn are values of veriate with frequencies f1, f2, f3,.........fn then their A.M. is given by

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + f_3x_3 + ...f_nx_n}{f_1 + f_2 + f_3 + ... + f_n} = \frac{\sum_{i=1}^{n} f_i x_i}{N}, \text{ where } N = \sum_{i=1}^{n} f_i$$

2. Properties of Arithmetic Mean :

(i) Sum of deviation of variate from their A.M. is always zero that is $\sum (x_i - \bar{x}) = 0$.

(ii) Sum of square of deviation of variate from their A.M. is minimum that is $\sum (x_i - \bar{x})^2$ is minimum

(iii) If $\bar{x}$ is mean of variate xi then

A.M. of $(x_i + \lambda) = \bar{x} + \lambda$

A.M. of $\lambda_i . x_i = \lambda . \bar{x}$

A.M. of $(ax_i + b) = a\bar{x} + b$

3. Me di an

The median of a series is values of middle term of series when the values are written is ascending order or descending order. Therefore median, divide on arranged series in two equal parts

For ungrouped distribution : If n be number of variates in a series then

$$\text{Median} = \begin{cases} \left(\frac{n+1}{2}\right)^{th} \text{term, when n is odd} \\ \text{Mean of } \left(\frac{n}{2}\right)^{th} \text{and} \left(\frac{n}{2}+1\right)^{th} \text{term when n is even} \end{cases}$$

4. **Mode**

If a frequency distribution the mode is the value of that variate which have the maximum frequency. Mode for

For ungrouped distribution :

The value of variate which has maximum frequency.

For ungrouped frequency distribution :

The value of that variate which have maximum frequency.
Relationship between mean, median and mode.

(i) In symmetric distribution, mean = mode = median

(ii) In skew (moderately asymmetrical) distribution,
median divides mean and mode internally in 1 : 2 ratio.

$$\Rightarrow \quad \text{median} = \frac{2(\text{Mean}) + (\text{Mode})}{3}$$

5. **Range**

$$\frac{\text{difference of extreme values}}{\text{sum of extreme values}} = \frac{L - S}{L + S}$$

where L = largest value and S = smallest value

6. **Mean deviation :**

$$\text{Mean deviation} = \sum_{i=1}^{n} \frac{|x_i - A|}{n}$$

$$\text{Mean deviation} = \sum_{i=1}^{n} \frac{f_i\,|x_i - A|}{N} \quad \text{(for frequency distribution)}$$

7. **Variance** :

Standard deviation = $+\sqrt{\text{variance}}$

formula

$$\sigma_x^2 = \frac{\sum (x_i - \bar{x})^2}{n}$$

$$\sigma_x^2 = \frac{\sum_{i=1}^{n} x_i^2}{n} - \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2 = \frac{\sum_{i=1}^{n} x_i^2}{n} - \left(\bar{x}\right)^2$$

$$\sigma_d^2 = \frac{\sum d_i^2}{n} - \left(\frac{\sum d_i}{n}\right)^2$$, where $d_i = x_i - a$, where a = assumed mean

(ii) coefficient of S.D. = $\left(\frac{\sigma}{\bar{x}}\right)$

coefficient of variation = $\left(\frac{\sigma}{\bar{x}}\right) \times 100$ (in percentage)

Properties of variance :

(i) $var(x_i + \lambda) = var(x_i)$

(ii) $var(\lambda . x_i) = \lambda^2 (var\ x_i)$

(iii) $var(a x_i + b) = a^2 (var\ x_i)$

where λ, a, b are constant.

MATHEMATICAL REASONING

Let p and q are statements

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$	$q \leftrightarrow p$
T	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	F	F	T	F	F	F
F	F	F	F	T	T	T	T

Tautology : This is a statement which is true for all truth values of its components. It is denoted by t.

Consider truth table of p v ~ p

p	~p	p v ~p
T	F	T
F	T	T

Fallacy : This is statement which is false for all truth values of its components. It is denoted by f or c. Consider truth table of p ˆ^~ p

p	~p	p⊠~p
T	F	F
F	T	F

(i)

Statements	p⊠q	p□q	p ⊠ q	p ⊠ q
Negation	(~p)⊠(~q)	(~p)□(~q)	p ⊠ (~ q)	p ⊠ –q

(ii)

Let p ⊠ q Then
(Contrapositive of p⊠ q) is (~q □ ~p)

SETS AND RELATION

Laws of Algebra of sets (Properties of sets):

(i) Commutative law : (A ⊠ B) = B ⊠ A ; A ⊠ B = B ⊠ A

(ii) Associative law: (A ⊠ B) ⊠ C=A ⊠ (B ⊠ C) ; (A ⊠ B)⊠C = A ⊠ (B ⊠ C)

(iii) Distributive law :

A ⊠⊠(B ⊠ C) = (A ⊠ B) ⊠ (A ⊠ C) ; A ⊠ (B ⊠ C) = (A ⊠ B) ⊠ (A ⊠ C)

(iv) De-morgan law : (A ⊠ B)' = A' ⊠ B' ; (A ⊠ B)' = A' ⊠ B'

(v) Identity law : A ⊠ U = A ; A ⊠ ⊠ = A

(vi) Complement law : A ⊠ A' = U, A ⊠ A' = ⊠, (A')' = A

(vii) Idempotent law : A ⊠ A = A, A ⊠ A = A

Some important results on number of elements in sets :

If A, B, C are finite sets and U be the finite universal set then

(i) n(A ⊠ B) = n(A) + n(B) – n(A ⊠ B)

(ii) n(A – B) = n(A) – n(A ⊠ B)

(iii) n(A ⊠ B ⊠ C) = n(A) + n(B) + n(C) – n(A ⊠ B) – n(B ⊠ C) – n (A ⊠ C) + n(A ⊠ B ⊠ C)

(iv) Number of elements in exactly two of the sets A, B, C

= n(A ⊠ B) + n(B ⊠ C) + n(C ⊠ A) – 3n(A ⊠ B ⊠ C)

(v) Number of elements in exactly one of the sets A, B, C

= n(A) + n(B) + n(C) – 2n(A ⊠ B) – 2n(B ⊠ C) – 2n(A ⊠ C) + 3n(A ⊠ B ⊠ C)

Types of relations :

In this section we intend to define various types of relations on a given set A.

(i) Void relation : Let A be a set. Then ⊠ ⊠ A × A and so it is a rela on A. This relation is called the void or empty relation on A.

(ii) Universal relation : Let A be a set. Then A × A ⊠ A × A and so it is a relation on A. This relation is called the universal relation on A.

(iii) Identity relation : Let A be a set. Then the relation I A = {(a, a) : a ⊠ A} on A is called the identity relationon A. In other words, a relation I A on A is called the identity relation if every element of A is related to itself only.

(iv) Reflexive relation : A relation R on a set A is said to be reflexive if every element of A is related to itself. Thus, R on a set A is not reflexive if there exists an element a ⊠ A such that (a, a) ⊠ R.

Note : Every identity relation is reflexive but every reflexive relation in not identity.

(v) Symmetric relation : A relation R on a set A is said to be a symmetric relation iff (a, b) ⊠ R ⊠ (b ,a) ⊠ R for all a, b ⊠ A. i.e. a R b ⊠ b R a for all a, b ⊠ A.

(vi) Transitive relation : Let A be any set. A relation R on A is said to be a transitive relation iff (a, b) ⊠ R and (b, c) ⊠ R ⊠ (a, c) ⊠ R for all a, b, c ⊠ A i.e. a R b and b R c ⊠ a R c for all a, b, c ⊠ A

(vii) Equivalence relation : A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. (a, a) ⊠ R for all a ⊠ A

(ii) it is symmetric i.e. (a, b) ⊠ R ⊠ (b, a) ⊠ R for all a, b ⊠ A

(iii) it is transitive i.e. (a, b) ⊠ R and (b, c) ⊠ R ⊠ (a, c) ⊠ R for all a,b⊠A